Additional Polynomial Practice and Exponents

Simplifying expressions

Expressions are like English nouns. They represent numbers. A single expression can be composed of many terms. Terms are separated from one another by addition or subtraction. For example, $4x + 11$ is an expression composed of two terms. However, the expression $3x$ has only one term. Nevertheless, we may readily identify two factors, namely 3 and x from the expression $3x$. Just remember that "factor" is a different distinction than "term."

This year, we saw that we can simplify algebraic expressions through addition and subtraction of like terms, those terms who have the same variable with the same exponent.

For example x^2 and $2x^2$ are like terms (with different coefficients); however, x^2 and x are not like terms.

Consequently, we may simplify the expression $x^2 + 2x^2$ by adding the terms together to get $3x^2$. But on the other hand, $x^2 + x$ is in simplest form. We cannot simplify the expression further through addition.

The following examples will require you to simplify expressions.

When combining like terms, the computation requires us to add their coefficients. Coefficients can be positive or negative, integers, rational and even irrational numbers.

Some problems will ask you to subtract one polynomial from another. Remember that when doing so, we are subtracting **every** term in parentheses.

We can consider this concept numerically first.

For example $-(7 + 2) = -(9) = -9$.

But the negative sign can be considered as a type of negation. We are negating (or taking the opposite of) the sum of 7 and 2. There was that funny phrase we had. The opposite of a sum is the sum of the opposites (of the addends).

Here the opposites of the addends are -7 and -2, and their sum is $-7 + (-2) = -9$, which is the same conclusion we arrived at just before. We are applying the same idea to algebraic terms.

As a final note here, we might consider this from another interpretation, that of distribution.

Remember that if we multiply any number by 1, the product is that number. If we multiply any number by -1, the product is the opposite of that number. With this in mind, we see that

$-(7 + 2) = -1(7 + 2)$.

Thus we might distribute -1 to 7 and then to 2 to get $-(7 + 2) = -1(7 + 2) = -1(7) + (-1)(2) = -7 + (-2) = -9$.

Simplify the following expressions using the distributive property and by combining like terms.

1.) $(3x^2 + 9x - 11) + (4x^2 - 7x + 23)$

2.) $(12 - 4x + 2x^2) - (5x^2 + 11x + 12)$

3.) $(-8x^2 - 4x + 11) - (5x^2 - 12x + 13 - y)$

4.) $9y^2 + 33 - 11y - (5y^2 + 23y - 12)$

5.) $ab^2 + 4b^2 - 7a^2 + (7a^2 + 11b^2 - 9b^2)$

6.) $(x - 4) - (a + \frac{1}{3}x + \frac{7}{4})$

7.)$\frac{1}{2}x - 7x + 9 + 7\frac{5}{7}x - 2\frac{1}{4}$

8.) $(12y^2 + 12y - 24) - 5(6y - 7y^2 + 12)$

9.) Challenge: $\frac{x+2}{4} - \frac{3x-8}{5} + \frac{x-5}{8}$

10.) $8y^2 + 8h - 12 + \frac{1}{2}(7y^2 - 4y + 18)$

11.) $3.5(d^2 + 9d - 12) - 2(8d^3 - 5d + 11)$

12.) $\frac{4}{5}f - 6f^2 + 12 + 6\frac{1}{3}f + 11.34 - \frac{14}{19}f^2$

13.) $2(x^2 - 8x + 14) - 9(3x^2 + 12x - 14)$

14.) $g - 14g^2 + 17 - 8g + 18g^2 - 14$

15.) $22x^2 + 43x - 14 - 56x^2 + 32x + 18$

16.) $\frac{2x-8y}{4}$

17.) Challenge: $\frac{3x+18y}{9} - \frac{27x+36y}{18}$

18.) $2(x^2 + 7x - 3) - (x^2 + 11x - 34)$

19.) $8x^2 + 4x - 12 - 3(-6x^2 - 11x + 12)$

20.) $-9y^2 - 7y - 34 - 9(-2y^2 - 8y + 25)$

Multiplying Monomials

Recall. A monomial is a term that has no variables in the denominator and whose variables have non-negative integer exponents.

An expression composed of unlike terms can be further simplified when the operation that relates them is multiplication (or division, which can be viewed as multiplying by reciprocals).

Find the following products.(Using the Laws of Exponents might be helpful! ... you can find them at the end of this review.)

1.) $3 \cdot x^2$

2.) $4 \cdot 9xy$

3.) $5s(s^2 \cdot 9t)$

4.) $\frac{21}{2}x \cdot \frac{2}{21}xy^2$

5.) $2x^4y^3(5x^5y^7)$

6.) $\frac{2}{3}h \cdot \frac{4}{5}h^2t$

7.) $\frac{x \cdot 2}{6} \cdot \frac{9}{11}$

8.) $\frac{2x \cdot 8y}{4} \cdot \frac{3x \cdot 4y}{12}$

9.) $\frac{2}{9}s^2t^3z^8 \cdot (4c \cdot 7s^5)$

10.) $xyz^7 \cdot x^5yz^7$

Multiplying Monomials and Binomials

Recall. A binomial is the sum of two monomials.

1.) $2(v+8)$

2.) $x(v+8)$

3.) $v(v+8)$

4.) $v^2(v+8)$

5.) $\frac{1}{4}(v+8)$

6.) $2x^2(x+t)$

7.) $3x^2y(7x^3+8)$

8.) $\frac{2}{3}(8d-12)$

9.) $-9(t+12)$

10.) $-12(-3r^2+1)$

11.) $33x(-4x^2+x)$

12.) $(9-y)(2x^3y^2)$

13.) $(8-3d)(2d^4)$

14.) $\frac{-1}{2}(5s+4)$

15.) $\frac{3}{5}(g+2)$

Multiplying Monomials and Polynomials

Recall. A polynomial is the sum of monomials.

1.) $\frac{3}{10}(g^2+9g-11)$

2.) $\frac{2}{5}(2t^3+4t^2-5t+12)$

3.) $(2x^2y^3)(4x^3y^4-12x^3t^2+12.5x^3y^9)$

4.) $-8y^2z^5(3\frac{3}{4}x+4x^2y^3z^3+xyz-1)$

5.) $3y^3(2+4x^2-7xy^2-12x^2y^8)$

Multiplying Binomials

When multiplying binomials, we are applying the distributive property twice. We may consider this type of process in quite a few ways. It becomes more challenging when we multiply a difference and a sum or when we multiply two differences together. This is largely due to the dual nature/interpretation of a plus or minus sign as a symbol of positivity or negativity and an operation symbol (addition or subtraction). Please consider the following. ...

You may skip what is in brackets and just read the text outside braces for a shorter version (like a gospel reading).

To write the following product as a sum of two terms, we need to distribute the 8 to the x and to the 2. This gives us $8(x+2) = 8(x) + 8(2) = 8x + 16$.

Notice that our "new" expression maintains the operation that separated x and 2 of the original expression. Let us also consider $4(9-y) = 4(9)-4(y)$. Again, the operation separating (or connecting really) the first and second terms is still subtraction.

Let us now consider the product of two sums. (Remember: A sum or difference can act as a factor!)

$(x + 4)(x + 3)$. We can first distribute x to each term of the second binomial and then distribute $+4$ to each term of the second binomial.

We have $(x + 4)(x + 3) = x(x) + x(3) + 4(x) + 4(3)$. If we consider an additional step, we can see more clearly which signs are being maintained as operation signs and which as negative/positive signs.

Let $y = x + 3$. Thus $(x + 4)(x + 3) = (x + 4)y$ And so, we have a singular distribution problem.

$(x + 4)y = x(y) + 4(y)$. Re-substituting $x + 3$ for y gives us $x(y) + 4(y) = x(x + 3) + 4(x + 3)$, and we have two singular distribution problems.

$x(x + 3) + 4(x + 3) = x(x) + x(3) + 4(x) + 4(3) = x^2 + 3x + 4x + 12 = x^2 + 7x + 12.$

{ Now let's look at some examples with differences.

$(x+7)(x-2)$. Let $t = x-2$. Then $(x+7)(x-2) = (x+7)t = x(t)+7(t) = x(x-2)+7(x-2) = x(x-2)+[7(x-2)]$. I have added the brackets because I am treating the plus sign before the 7 as an addition symbol. In particular, we are adding the product of 7 and $(x-2)$ to $x(x-2)$.

$x(x-2) + [7(x-2)] = x(x) - x(2) + [7(x) - 7(2)] = x^2 - 2x + (7x - 14) = x^2 - 2x + 7x - 14 = x^2 + 5x - 14.$

Let's consider $(x - 2)(x + 5)$. Let $a = x + 5$. Then $(x - 2)(x + 5) = (x - 2)a = x(a) - 2(a) = x(x + 5) - [2(x + 5)]$. I have added the brackets because I am treating the minus sign as an operation. In particular, we are subtracting the <u>product</u> of 2 and $(x + 5)$ from $x(x + 5)$.

$x(x + 5) - [2(x + 5)] = x(x) + x(5) - [2(x) + 2(5)] = x^2 + 5x - (2x + 10) = x^2 + 5x - 2x - 10 = x^2 + 3x - 10.$}

Now, here is a final way that I will offer to consider multiplying binomials.

When multiplying a difference and a sum or when multiplying two differences, express the difference as a sum (subtracting is the same as adding the negative!) This way you may separate each term by addition (at least prior to simplification).

For example $(x-2)(x+7) = (x+-2)(x+7) = x(x)+x(7)+(-2)(x)+(-2)(7) = x^2+7x+-2x+-14 = x^2+7x-2x-14 = x^2 + 5x - 14$.

$(x + 9)(x - 3) = (x + 9)(x + -3) = x(x) + x(-3) + 9(x) + 9(-3) = x^2 + -3x + 9x + -27 = x^2 + 6x - 27.$

Find the following products.

1.) $(x-11)(x+2)$

2.) $(x+4)(x+3)$

3.) $(y+9)(y-12)$

4.) $(11+t)(5+t)$

5.) $(x-7)(x-9)$

6.) $(2x+1)(3x-11)$

7.) $(2c+3)(c-4)$

8.) $(g-8)(g+11)$

9.) $(f+12)(f+5)$

10.) $(r-3)(r+5)$

11.) $(s+7)(s-8)$

12.) $(4+w)(3-w)$

13.) $(8-t)(6+t)$

14.) $(3q+11)(4q+4)$

15.) $(6d-11)(5d+7)$

16.) $(a-3)(a-2)$

17.) $(x^2+2)(x^2+3)$

18.) $(x+y)(x+y)$

19.) $(a-b)(a+b)$

20.) $(x-4)(x+4)$

21.) $(x-2)^2$

22.) $(x+3)^2$

Multiplying Binomials and Trinomials

Recall. A trinomial is the sum of three monomials.

Now instead of distributing each term of our binomial to two terms, we must distribute each term of our binomial to three terms. Or, we may also consider applying the distributive property three times, that is, distributing each term of our trinomial to each term of the binomial.

Find the following products.

1.) $(x+5)(x^2+7x-11)$

2.) $(x-3)(4x^2-9x+3)$

3.) $(x-7)(7xy+8y^2-12)$

4.) $(8-t)(13t+4t^2-22)$

5.) $(2c+4)(3rs^2-4rt^3+5c^7t)$

Factoring out "algebraic" GCF

If we consider the numbers 18 and 15, we can find the respective prime factorizations of both. We have that $18 = 2\cdot3\cdot3 = 2\cdot3^2$, and $15 = 3 \cdot 5$. Thus, their greatest common factor is 3. Notice that their GCF is not 3^2. Only 3, not 9, is a factor of both 18 and 15. GCF takes into account not only the values of a number's factors but their "frequency" if you will.

Variables represent numbers, and so we may extend this concept of GCF to algebraic terms. Let us consider an example.

Let us consider specifically the terms $2xy^2$ and $8x^4y^5$. We will first break down $2xy^2$ into its factors. We have $2xy^2 = 2 \cdot x \cdot y^2 = 2 \cdot x \cdot y \cdot y$.

Likewise, $8x^4y^5 = 2 \cdot 2 \cdot 2 \cdot x^4 \cdot y^5 = 2^3 \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y = 2^3 \cdot x^3 \cdot x \cdot y^2 \cdot y \cdot y \cdot y = 2 \cdot 2^2 \cdot x \cdot x^3 \cdot y^2 \cdot y^3$. We might notice that their greatest common factor, the product of all the prime factors they share, is $2xy^2$.

Now, let us consider their sum $2xy^2 + 8x^4y^5$.

Recall the distributive property. We have that $a(b+c) = ab + ac$. Now, there exists a property called the Symmetric Property. It states that if $x = y$, then $y = x$. And so, if $a(b+c) = ab + ac$, then $ab + ac = a(b+c)$. In fact, because of the symmetric property, the fact that $ab + ac = a(b+c)$ is also justified by the Distributive Property of Multiplication over Addition. However, you might often hear of "factoring" as applying the distributive property "in reverse."

If we wanted to "factor out" the GCF of $2xy^2+8x^4y^5$, we would be factoring out $2xy^2$. But, we want to represent $2xy^2+8x^4y^5$ with an equivalent expression. We have to apply the distributive property, but this time, we are asking ourselves what factors are "left."

If we consider $8x^4y^5 = 2 \cdot 2^2 \cdot x \cdot x^3 \cdot y^2 \cdot y^3$, then after "taking out" the $2, x$, and y^2, we are left with the factors $2^2, x^3$, and y^3.

We might also consider $2xy^2 = 2 \cdot x \cdot y^2$, then after "taking out," the $2, x$, and y^2, what are we left with? Well, one is a factor of every term, so technically we are left with one. This will be very important in just a minute!

We write out the GCF first and then put the sum of the respective products of the remaining factors of each term in parentheses. We should still have two terms in parentheses, and still separated by addition in this case (as we have not factored out a negative number). We have $2xy^2 + 8x^4y^5 = 2xy^2(1 + 2^2x^3y^3) = 2xy^2(1 + 4x^3y^3)$.

You might notice that if we distribute $2xy^2$ to each of these terms in parentheses, we arrive at our original expression.

Factor out the GCF of the following polynomials to express their sum as a product.

1.) $6x^2y^3 + 4xy^8 - 12xy^5 + 2xy^3$

2.) $10gh - 21g^2h^2 + 20g^3h^3 - 30g^4h^4$

3.) $5x^5y^7 - 10x^4y^8 + 5x^2y^2 - 10x^8y^6$

4.) $3s^2t^3 + 9s^4t^5 - 12s^3t^6 - 15s^2$

5.) $4 - 16xy^2 + 24x^2y - 36x^3y^8$

Factoring Trinomials

Let us think about binomials again. Let us consider two binomials, namely $x + a$ and $x + b$, where a and b are constants.

If we multiply them together, we get $(x + a)(x + b) = x^2 + bx + ax + ab = x^2 + (b + a)x + ab = x^2 + (a + b)x + ab$.

Similar to the previous section, we might understand factoring as applying the distributive property in reverse. But now, we are interested in, not factoring out the GCF of these three terms, but in factoring it into two binomials. See that in the example above $(x + a)$ and $(x + b)$ are factors of one another.

If given the trinomial, $x^2 + cx + d$, we want two binomials, call them $x + a$ and $x + b$, such that $a + b = c$ and $ab = d$. See the second paragraph for why.

If we are given the trinomial $x^2+7x+12$, we are looking for binomials of the form $x+a$ and $x+b$, such that $a+b = 7$ and $ab = 12$.

Let us start with factors of 12.

We have that $12 = 2 \cdot 6 = -2 \cdot -6 = 3 \cdot 4 = -3 \cdot -4$. Notice though, that of these options, $3 + 4 = 7$. Therefore $a = 3$ and $b = 4$. (We could also have said that $a = 4$ and $b = 3$.)

And so, our factorization of $x^2 + 7x + 12$ is $x^2 + 7x + 12 = (x + 30(x + 4)$.

Factor the following trinomials into the product of two binomials.

1.) $x^2 + 8x + 15$

2.) $x^2 + 4x + 4$

3.) $x^2 + 21x + 110$

4.) $x^2 - 7x + 12$

5.) $x^2 - x - 30$

6.) $x^2 + x - 56$

7.) $x^2 - 4$ ... Here $a + b = 0$.

8.) $x^2 + 6x + 9$

Challenge

Before the coefficient of our x^2 term was 1. Now, when the coefficient of x^2 is not 1, we cannot use the same procedure as before. We will have to slightly alter it.

9.) $2x^2 + 5x + 2$

10.) $3x^2 - 10x - 8$

Exponents Review

Recall in the expression a^x, a is called the base, x is called the exponent and together a^x is referred to as a power. Your base and exponent can be fractions and/or negative.

Important Reminder: $-3^2 = -9$ but $(-3)^2 = 9$. In the first expression, 3 is the base, whereas in the second expression -3 is the base.

1.) 4^3

2.) -5^8

3.) $(-9)^3$

4.) $-(\frac{1}{2})^3$

5.) -2^3

6.) $\frac{5^2}{6^2}$

Exponents in Algebraic Expressions

Please note. Many of the problems in this section push the concepts we learned about exponents further and are challenging. Try your hand at it!

Let us review our Laws of Exponents.

Let x be a real number, then $x^0 = 1$. This is in keeping with the way we define exponents because
$2^1 = 2$,
$2^2 = 4$,
$2^3 = 8$,
$2^4 = 16$, etc. "Working backwards," we might notice that $16 \div 2 = 8, 8 \div 2 = 4, 4 \div 2 = 2$. Going one step further back, we have $2 \div 2 = 1$. And so we might be content with $2^0 = 1$.

Law of Exponents for Products

Let $x, a,$ and b be real numbers, then $x^a \cdot x^b = x^{a+b}$.

Notice here that the exponent of x is a sum.

*Note: $x^a \cdot y^b \neq (xy)^{a+b}$. Why?

Law of Exponents for Powers

Let $x, a,$ and b be real numbers, then $(x^a)^b = x^{ab}$.

Notice here that the exponent of x is a product. Compare this to the exponent of x when we applied the Law of Exponents for Products.

Law of Exponents for Quotients

Let $x, a,$ and b be real numbers, then $\frac{x^a}{x^b} = x^{a-b}$.

Negative Exponents

Now, let's consider $\frac{x^4}{x^5}$. Applying the Law of Exponents for Quotients, we have $\frac{x^4}{x^5} = x^{-1}$. But what does that mean? Well, let's first consider this in terms of our "normal" rules of arithmetic.

We have that $\frac{x^4}{x^5} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x}$. Because our numerator and denominator are comprised only of factors, we may cancel quite freely. In fact, after cancelling out all the common factors, we are left with $\frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x}$. This must mean, $x^{-1} = \frac{1}{x}$. In fact, this is actually how we define negative exponents.

Let x and a be real numbers, then $x^{-a} = \frac{1}{x^a}$.

Simplify the following expressions. Please do not leave answers with negative exponents.

1.) $x^2 \cdot x^{18}$

2.) $2x^5 \cdot 7x^9$

3.) $x^7 \div x^4$

4.) $\frac{x^{23}}{x^{11}}$

5.) $\frac{a^3}{a^9}$

6.) $9ab^0 \cdot a^5b^3$

7.) $\frac{ab^3}{b^3}$

8.) $5x^2y^{14}(9x^8y^4)$

9.) $(x^3)^2$

10.) $(y^3)^5$

11.) $3^8 \cdot 3^4$

12.) $\frac{(s^{11})^2}{s^5}$

13.) $x^4y^3 \cdot 8x^2y^4$

14.) $\frac{ab^3z^2}{(z^2)^5}$

15.) Is $(a+b)^2 = a^2 + b^2$?