Triangle $CDE$ has vertices $C(-5, -1)$, $D(-2, -5)$, and $E(-1, -1)$. Graph $\triangle CDE$ and its image after the indicated glide reflection.

1. Translation: along $(4, 0)$ Reflection: in $x$-axis

**SOLUTION:**
Translation along $(4, 0)$:

$$(x, y) \Rightarrow (x + 4, y)$$

$C(-5, -1) \Rightarrow C'(1, -1)$

$D(-2, -5) \Rightarrow D'(2, -5)$

$E(-1, -1) \Rightarrow E'(3, -1)$

**ANSWER:**

Reflection in $x$-axis:

$$(x, y) \Rightarrow (x, -y)$$

$C'(-1, -1) \Rightarrow C''(-1, 1)$

$E'(3, -1) \Rightarrow E''(3, 1)$

$D'(2, -5) \Rightarrow D''(2, 5)$

Graph $\triangle CDE$ and its image.

2. Translation: along $(0, 6)$ Reflection: in $y$-axis

**SOLUTION:**
Translation along $(0, 6)$:

$$(x, y) \Rightarrow (x, y + 6)$$

$C(-5, -1) \Rightarrow C''(-5, 5)$

$D(-2, -5) \Rightarrow D'(2, 1)$

$E(-1, -1) \Rightarrow E'(1, 5)$

**Reflection in $y$-axis:**

$$(x, y) \Rightarrow (-x, y)$$

$C''(-5, 5) \Rightarrow C''(5, 5)$

$D'(2, 1) \Rightarrow D''(2, 1)$

$E'(1, 5) \Rightarrow E''(1, 5)$

Graph $\triangle CDE$ and its image after the indicated glide reflection.
Graph \( \triangle CDE \) and its image.

Rotation of 90° about the origin:
\((x, y) \Rightarrow (-y, x)\)

\(J' (2, -5) \Rightarrow J'' (5, 2)\)

\(K' (6, -5) \Rightarrow K'' (5, 6)\)

The graph \(JK\) and its image is given below.

Copy and reflect figure \(S\) in line \(m\) and then line \(m'\).
4. **SOLUTION:**

**Step 1:** Reflect \( A \) in line \( m \).

![Diagram](triangle.png)

**Step 2:** Reflect \( B \) in line \( p \).

![Diagram](triangle.png)

By Theorem 9.2, the composition of two reflections in parallel vertical lines \( m \) and \( p \) is equivalent to a horizontal translation right \( 2 \cdot 1.5 \) or 3 inches.

**ANSWER:**

![Diagram](triangle.png)

horizontal translation 3 in. to the right

---

5. **SOLUTION:**

**Step 1:** Reflect \( S \) in line \( m \).

![Diagram](triangle.png)

**Step 2:** Reflect \( S \) in line \( p \).

![Diagram](triangle.png)

By Theorem 9.3, the composition of two reflections in intersecting lines \( m \) and \( p \) is equivalent to a 2 \cdot 50^\circ \) or 100^\circ \) clockwise rotation about the point where lines \( m \) and \( p \) intersect.

**ANSWER:**

![Diagram](triangle.png)

rotation clockwise 100^\circ \) about point where lines \( m \) and \( p \) intersect
Determine the preimage given the image and the composition of transformations.
6. reflection in line \(j\), translation along \(\overrightarrow{n}\)

**SOLUTION:**
First, reflect \(\triangle GHF\) over line \(j\), using a ruler

Then, translate \(\triangle G'H'F'\) along vector \(\overrightarrow{n}\).

**ANSWER:**

SOLUTION:
First, rotate Quadrilateral \(BCDE\) 180° about the origin. This can be done by using the rule
\[(x, y) \rightarrow (-x, -y)\]
\[(x, y) \rightarrow (-x, -y)\]
\[B(-4, -3) \rightarrow B'(4, 3)\]
\[C(-3, -1) \rightarrow C'(3, 1)\]
\[D(-1, -1) \rightarrow D'(1, 1)\]
\[E(-1, -3) \rightarrow E'(1, 3)\]

Then translate \(B'C'D'E'\) 4 units down, using the rule
\[(x, y) \rightarrow (x, y - 4)\]
\[(x, y) \rightarrow (x, y - 4)\]
\[B'(4, 3) \rightarrow B''(4, -1)\]
\[C''(3, 1) \rightarrow C''(3, -3)\]
\[D'(1, 1) \rightarrow D''(1, -3)\]
\[E'(1, 3) \rightarrow E''(1, -1)\]

**ANSWER:**

7. rotation 180° about origin, translation 4 units down
8. **TILE PATTERNS** Viviana is creating a pattern for the top of a table with tiles in the shape of isosceles triangles. Identify the sequence of transformations that was used to transform the white triangle to the blue triangle.

![Triangle](image)

**SOLUTION:**

**Step 1:** Reflect the left white triangle with the base as the line of reflection.

![Reflect](image)

**Step 2:** Translate the triangle to the right.

![Translate](image)

The resulting transformation can also be called a glide reflection.

**ANSWER:**

reflection and translation (glide reflection)

Graph each figure with the given vertices and its image after the indicated glide reflection.

9. \(\triangle RST: R(1, -4), S(6, -4), T(5, -1)\)

Translation: along \((2, 0)\)

Reflection: in \(x\)-axis

**SOLUTION:**

First, reflect \(\triangle RST\) over the \(x\)-axis to make \(\triangle R'S'T'\). Then, translate \(\triangle R'S'T'\) two units to the right to make \(\triangle R''S''T''\).
3-4 Compositions of Transformations

10. $\triangle JKL$: $J(1, 3), K(5, 0), L(7, 4)$
   Translation: along $\langle -3, 0 \rangle$
   Reflection: in x-axis

   **SOLUTION:**
   First, translate $\triangle JKL$ three units to the left to make $\triangle J'K'L'$. Then, reflect $\triangle J'K'L'$ in the x-axis to make $\triangle J''K''L''$.

   ![Diagram of $\triangle JKL$ and its transformations](attachment:triangle_jkl_transformations.png)

   **ANSWER:**
   ![Diagram of transformed triangle](attachment:transformed_triangle.png)

11. $\triangle XYZ$: $X(-7, 2), Y(-5, 6), Z(-2, 4)$
   Translation: along $\langle 0, -1 \rangle$
   Reflection: in y-axis

   **SOLUTION:**
   First, translate $\triangle XYZ$ one unit down to make $\triangle X'Y'Z'$. Then, reflect $\triangle X'Y'Z'$ in the y-axis to make $\triangle X''Y''Z''$.

   ![Diagram of $\triangle XYZ$ and its transformations](attachment:triangle_xyz_transformations.png)

   **ANSWER:**
   ![Diagram of transformed triangle](attachment:transformed_triangle_xy.png)
3-4 Compositions of Transformations

12. \( \triangle ABC; A(2, 3), B(4, 7), C(7, 2) \)
   Translation: along \( \{0, 4\} \)
   Reflection: in \( y\)-axis

   **SOLUTION:**
   First, translate \( \triangle ABC \) four units up to make \( \triangle A'B'C' \).
   Then, reflect \( \triangle A'B'C' \) in the \( y\)-axis to make \( \triangle A''B''C'' \).

   ![Diagram of \( \triangle ABC \) and \( \triangle A''B''C'' \)]

   **ANSWER:**

13. \( \overline{WX}; W(-4, 6) \) and \( X(-4, 1) \)
   Reflection: in \( x\)-axis Rotation: 90° about origin

   **SOLUTION:**
   Reflection in \( x\)-axis:
   \( (x, y) \mapsto (x, -y) \)
   \( W(-4, 6) \mapsto W'(-4, -6) \)
   \( X(-4, 1) \mapsto X'(-4, -1) \)

   Rotation of 90° about the origin:
   \( (x, y) \mapsto (-y, x) \)
   \( W'(-4, -6) \mapsto W''(6, -4) \)
   \( X'(-4, -1) \mapsto X''(1, -4) \)

   ![Diagram of \( W \) and \( X \) with reflected and rotated points]

   **ANSWER:**
14. \( \overline{AB} : A(-3, 2) \) and \( B(3, 8) \) Rotation: 90° about origin
   Translation: along \( \langle 4, 4 \rangle \)

   **SOLUTION:**
   Rotation of 90° about the origin:
   \((x, y) \Rightarrow (-y, x)\)
   \(A(-3, 2) \Rightarrow A'(-2, -3)\)
   \(B(3, 8) \Rightarrow B'(-8, 3)\)
   
   Translation along \( \langle 4, 4 \rangle : \)
   \((x, y) \Rightarrow (x + 4, y + 4)\)
   \(A'(-2, -3) \Rightarrow A''(2, 1)\)
   \(B'(-8, 3) \Rightarrow B''(-4, 7)\)

   **Draw a graph.**

   **ANSWER:**

15. \( \overline{FG} : F(1, 1) \) and \( G(6, 7) \) Reflection: in x-axis
   Rotation: 180° about origin

   **SOLUTION:**
   Reflection in x-axis:
   \((x, y) \Rightarrow (x, -y)\)
   \(F(1, 1) \Rightarrow F'(1, -1)\)
   \(G(6, 7) \Rightarrow G'(6, -7)\)
   
   Rotation of 180° about the origin:
   \((x, y) \Rightarrow (-x, -y)\)
   \(F'(1, -1) \Rightarrow F''(-1, 1)\)
   \(G'(6, -7) \Rightarrow G''(-6, 7)\)

   **Draw a graph.**

   **ANSWER:**
3-4 Compositions of Transformations

16. \( \overline{RS} : R(2, -1) \) and \( S(6, -5) \) Translation: along \( \langle -2, -2 \rangle \) Reflection: in \( y \)-axis

**SOLUTION:**
Translation along \( \langle -2, -2 \rangle \):
\[(x, y) \Rightarrow (x - 2, y - 2) \]
\( R(2, -1) \Rightarrow R'(0, -3) \)
\( S(6, -5) \Rightarrow S'(4, -7) \)

Reflection in \( y \)-axis:
\[(x, y) \Rightarrow (-x, y) \]
\( R'(0, -3) \Rightarrow R''(0, -3) \)
\( S'(4, -7) \Rightarrow S''(-4, -7) \)

Draw a graph.

**ANSWER:**

Copy and reflect figure \( D \) in line \( m \) and then line \( p \). Then describe a single transformation that maps \( D \) onto \( D' \).

17. **SOLUTION:**

**Step 1:** Reflect \( D \) in line \( m \).

**Step 2:** Reflect \( D' \) in line \( p \).

By Theorem 9.2, the composition of two reflections in parallel vertical lines \( m \) and \( p \) is equivalent to a horizontal translation right 2 \( \cdot \) 2 or 4 centimeters.

**ANSWER:**

horizontal translation 4 cm to the right
**3-4 Compositions of Transformations**

**SOLUTION:**

**Step 1:** Reflect $D$ in line $m$.

**Step 2:** Reflect $D'$ in line $p$.

By Theorem 9.2, the composition of two reflections in parallel vertical lines $m$ and $p$ is equivalent to a vertical translation down $2 \cdot 1.2$ or 2.4 inches.

**ANSWER:**

vertical translation 2.4 in. down

**SOLUTION:**

**Step 1:** Reflect $D$ in line $m$.

**Step 2:** Reflect $D'$ in line $p$.

By Theorem 9.3, the composition of two reflections in intersecting lines $m$ and $p$ is equivalent to a $2 \cdot 35^\circ$ or $70^\circ$ clockwise rotation about the point where lines $m$ and $p$ intersect.

**ANSWER:**

$70^\circ$ rotation about the point where lines $m$ and $p$ intersect
SOLUTION:

Step 1: Reflect D in line m.

Step 2: Reflect D’ in line p.

By Theorem 9.3, the composition of two reflections in intersecting lines m and p is equivalent to a 2 • 105° or 210° counterclockwise rotation about the point where lines m and p intersect.

ANSWER:

210° rotation about the point where lines m and p intersect
3-4 Compositions of Transformations

22. Reflection in line $x = y$, rotation $90^\circ$ about the origin

SOLUTION:
Work backwards. First, rotate $\triangle ABC$ $90^\circ$ about the origin, using the rule $(x, y) \rightarrow (y, -x)$.

$(x, y) \rightarrow (y, -x)$
$A'' (-4, -4) \rightarrow A' (4, -4)$
$B'' (-4, -2) \rightarrow B' (4, -2)$
$C'' (-1, -1) \rightarrow C' (1, -1)$

Then, reflect $\triangle A'B'C'$ in the line $y = x$, using the rule $(x, y) \rightarrow (y, x)$.

$(x, y) \rightarrow (y, x)$
$A'(4, -4) \rightarrow A(4, -4)$
$B'(4, -2) \rightarrow B(4, -2)$
$C'(1, -1) \rightarrow C(1, -1)$

ANSWER:

MODELING Identify the sequence of transformations that will create the outlines kimono fabric patterns.

23. Refer to the image of kimono fabric on page 255.

SOLUTION:
From the red flower to the blue flower, it is a translation along the vector $\vec{k}$.

ANSWER:
translation

24. Refer to the kimono fabric pattern given on page 255.

SOLUTION:
From the blue butterfly to the red butterfly, it is a rotation about a point so that it is oriented in the correct direction, followed by a translation that moves it to the right place.

ANSWER:
rotation, then translation

25. Refer to the kimono fabric pattern given on page 255.

SOLUTION:
It is a reflection on a line that is nearly vertical between the two fans followed by a translation to put them in the same position.

ANSWER:
reflection, then translation
26. **MODELING** Elizabeth has airbrushed the pattern shown onto her skateboard. What combination of transformations did she use to create the pattern?

**SOLUTION:**
Method 1:
Step 1: Reflect figure in line \( m \).

\[
\begin{array}{c}
\text{Step 2: Reflect figure in line } p. \\
\end{array}
\]

To get the side figures, translate the back figure right the translate it up and down.

This is also know as a glide reflection.

Method 2:
You can create the patterns with translations only.

Translate the middle figure to the right.

To get the side figures, translate the back figure right the translate it up and down.

**ANSWER:**
glide reflection or two translations

**ALGEBRA** Graph each figure and its image after the indicated transformations.

27. Rotation: 90° about the origin Reflection: in \( x \)-axis

**SOLUTION:**
Identify two points on the line \( y = 3x + 1 \). Let \( A \) be at (0, 1) and \( B \) at (-2, -5).

**Step 1:** Rotate 90° about the origin.
\((x, y) \Rightarrow (-y, x)\)

\[A(0, 1) \Rightarrow A'(1, 0)\]

\[B(-2, -5) \Rightarrow B'(5, -2)\]

Find the equations using the two points.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
m = \frac{-2 - 0}{5 - (-1)}
\]

\[
m = \frac{-2}{6}
\]

\[
m = \frac{-1}{3}
\]

\[
y - y_1 = m(x - x_1)
\]

\[
y - (1) = -\frac{1}{3}(x - (-1))
\]

\[
y = -\frac{1}{3}x - \frac{1}{3}
\]

**Step 2:** Reflect line in \( x \)-axis.
3-4 Compositions of Transformations

$(x, y) \rightarrow (x, -y)$

$A'(1, 0) \rightarrow A''(1, 0)$

$B'(5, -2) \rightarrow B''(5, 2)$

Find the equations using the two points.

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{2 - 0}{5 - (-1)}$

$m = \frac{2}{6}$

$m = \frac{1}{3}$

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{1}{3}(x - (-1))$

$y = \frac{1}{3}x + \frac{1}{3}$

**SOLUTION:**

Select several points on the quadratic $y = (x - 3)^2$.

$A(3, 0), B(4, 1), C(5, 4), D(6, 9), E(0, 9), F(1, 4), G(2, 1)$

**Step 1:** Reflect the quadratic in the $x$-axis.

$(x, y) \rightarrow (x, -y)$

$A(3, 0) \rightarrow A'(3, 0)$

$B(4, 1) \rightarrow B'(4, -1)$

$C(5, 4) \rightarrow C'(4, -5)$

$D(6, 9) \rightarrow D'(6, -9)$

$E(0, 9) \rightarrow E'(0, -9)$

$F(1, 4) \rightarrow F'(1, -4)$

$G(2, 1) \rightarrow G'(2, -1)$

The equation of the reflected quadratic is $y = -(x - 3)^2$.

**Step 2:** Reflect the quadratic in the $y$-axis

3-4 Compositions of Transformations

\((x, y) \Rightarrow (\ -x, \ y)\)

\(A' (3, 0) \Rightarrow A'' (\ -3, \ 0)\)

\(B' (4, \ -1) \Rightarrow B'' (\ -4, \ -1)\)

\(C' (5, \ -4) \Rightarrow C'' (\ -4, \ -5)\)

\(D' (6, \ -9) \Rightarrow D'' (\ -6, \ -9)\)

\(E' (0, \ -9) \Rightarrow E'' (\ -0, \ -9)\)

\(F' (1, \ -4) \Rightarrow F'' (\ -1, \ -4)\)

\(G' (2, \ -1) \Rightarrow G'' (\ -2, \ -1)\)

The equation of the reflected quadratic is

\(y = -(x + 3)^2\).

ANSWER:

29. \(\triangle ABC\) is reflected in the \(x\)-axis and then rotated 180° about the origin. \(\triangle A'B'C'\) has vertices \(A''(3, 1)\), \(B''(2, 3)\), and \(C''(1, 0)\). Find the coordinates of \(\triangle ABC\).

SOLUTION:

Work backwards to find the coordinates of \(\triangle ABC\).

To rotate a point 180° clockwise back about the origin, multiply the \(x\)- and \(y\)-coordinate of each vertex by \(-1\).

\((x, y) \Rightarrow (\ -x, \ -y)\)

\(A'' (3, 1) \Rightarrow A' (\ -3, \ -1)\)

\(B'' (2, 3) \Rightarrow B' (\ -2, \ -3)\)

\(C'' (1, 0) \Rightarrow C' (\ -1, 0)\)

Then, reflect \(\triangle A'B'C'\) back in the \(x\)-axis

\((x, y) \Rightarrow (x, \ -y)\)

\(A' (\ -3, \ -1) \Rightarrow A (\ -3, 1)\)

\(B' (\ -2, \ -3) \Rightarrow B (\ -2, 3)\)

\(C' (\ -1, 0) \Rightarrow C (\ -1, 0)\)

Then \(A (\ -3, 1)\), \(B (\ -2, 3)\) and \(C (\ -1, 0)\).

ANSWER:

\(A (\ -3, 1), B (\ -2, 3), C (\ -1, 0)\)
30. **PROOF** Write a paragraph proof for one case of the Composition of Isometries Theorem.

Given: A translation along \( \langle a, b \rangle \) maps \( X \) to \( X' \) and \( Y \) to \( Y' \).

A reflection in \( z \) maps \( X' \) to \( X'' \) and \( Y' \) to \( Y'' \).

Prove: \( XY \equiv X''Y'' \)

\[ \text{SOLUTION:} \]
Walk through the proof step by step. Review what is given and what needs to be proved. Here, you are given two translations and two reflections. Show that the image and preimage are congruent. Use the properties that you have learned about reflections and translations to walk through the proof.

Proof: It is given that a translation along \( \langle a, b \rangle \) maps \( X \) to \( X' \) and \( Y \) to \( Y' \). Using the definition of a translation, points \( X \) and \( Y \) move the same distance in the same direction, therefore \( XY \equiv XY' \). It is also given that a reflection in \( z \) maps \( X' \) to \( X'' \) and \( Y' \) to \( Y'' \). Using the definition of a reflection, points \( X \) and \( Y \) are the same distance from line \( z \), so \( X'Y' \equiv X''Y'' \). By the Transitive Property of Congruence, \( XY \equiv X''Y'' \).

**ANSWER:**

Proof: It is given that a translation along \( \langle a, b \rangle \) maps \( X \) to \( X' \) and \( Y \) to \( Y' \). Using the definition of a translation, points \( X \) and \( Y \) move the same distance in the same direction, therefore \( XY \equiv XY' \). It is also given that a reflection in \( z \) maps \( X' \) to \( X'' \) and \( Y' \) to \( Y'' \). Using the definition of a reflection, points \( X'' \) and \( Y'' \) are the same distance from line \( z \) as \( X' \) and \( Y' \), so \( X'Y' \equiv X''Y'' \). By the Transitive Property of Congruence, \( XY \equiv X''Y'' \).

**ANSWER:**

31. **KNITTING** Tonisha is knitting a scarf using the tumbling blocks pattern shown at the right. Describe the transformations combined to transform the red figure to the blue figure.

**SOLUTION:**

**Step 1:** Reflect figure over top edge. \( Q \) reflects to \( Q' \) and \( S \) to \( S' \).

**Step 2:** Reflect figure over top edge. \( Q' \) reflects to \( Q'' \) and \( S' \) to \( S'' \).

Thus, a double reflection describes the transformation from the pink to blue figure.

**ANSWER:**

double reflection

**Identify the sequence of transformations that will carry the preimage to the final image.**

**SOLUTION:**

**Step 1:** Translate \( \triangle JKL \).
Describe the translations.
\[ J(2, 2) \Rightarrow J'(1, -4) \]
\[ K(3, 5) \Rightarrow K'(2, -1) \]
\[ L(6, 2) \Rightarrow L'(5, -4) \]

\[ x \text{ translation: } 5 - 6 = -1 \]
\[ y \text{ translation: } -4 - 2 = -6 \]

\[ \langle x, y \rangle = \langle x - 1, y - 6 \rangle \]

Thus, the translation vector is \( \langle -1, -6 \rangle \)

**Step 2:** Reflect \( \triangle J'K'L' \) in the \( y \)-axis.

Describe the transformation.
\[ J'(1, -4) \Rightarrow J''(-1, -4) \]
\[ K'(2, -1) \Rightarrow K''(-2, -1) \]
\[ L'(5, -4) \Rightarrow L''(-5, -4) \]

The transformation can be described by
\[ \langle x, y \rangle = \langle -x, y \rangle. \]

The transformation is a reflection in the \( y \)-axis.

**ANSWER:**
translation along \( (-1,-6) \) and reflection in the \( y \)-axis

**SOLUTION:**

**Step 1:** Rotate figure \( QRST \).
Describe the transformation.
\[ Q(-6, 5) \Rightarrow Q'(6, -5) \]
\[ R(-2, 6) \Rightarrow R'(2, -6) \]
\[ S(-1, 3) \Rightarrow S'(1, -3) \]
\[ T(-5, 2) \Rightarrow T'(5, -2) \]

Notice that the signs of \( x \)- and \( y \)-coordinates change.
The rotation can be described by
\[ \langle x, y \rangle \rightarrow \langle -x, -y \rangle. \]

Thus, the rotation is a \( 180^\circ \) rotation about the origin.

**Step 2:** Reflection \( Q'R'S'T' \).
3-4 Compositions of Transformations

Describe the transformation.

\( Q'(6, -5) \Rightarrow Q'(6, 5) \)

\( R'(2, -6) \Rightarrow R'(2, 6) \)

\( S'(1, -3) \Rightarrow S'(1, 3) \)

\( T'(5, -2) \Rightarrow T'(5, 2) \)

Notice that the signs of \( y \)-coordinates change.

Thus, the reflection can be describe by

\( (x, y) \rightarrow (x, -y) \).

Therefore, the reflection is in the \( x \)-axis.

**ANSWER:**

rotation \( 180^\circ \) about the origin and reflection in the \( x \)-axis

**34. PROOF** Write a two-column proof of Theorem 3.2.

Given: A reflection in line \( p \) maps \( \overline{BC} \) to \( \overline{B'C'} \).

A reflection in line \( q \) maps \( \overline{B'C'} \) to \( \overline{B''C''} \).

\( p \parallel q \), \( AD = x \)

**Prove:**

a. \( BB'' \perp p \), \( BB'' \perp q \)

b. \( BB'' = 2x \)

**SOLUTION:**

In the 1st row, state the given information about the reflections and the distance between \( p \) and \( q \).

In the 2nd row, use the definition of perpendicular bisectors to show \( p \) is the perpendicular bisector of \( BB' \), and \( q \) is the perpendicular bisector of \( B'B'' \).

In the 3rd row, use segment addition to show \( BB' + B'B'' = BB'' \).

In the 4th row, use the concept that a line perpendicular to a portion of a segment is perpendicular to the whole segment.

In the 5th row, apply the reflexive property.

In the 6th row, define the relationship between segment length using the definition of congruent segments.

In the 7th row, apply the segment addition postulate.

In the 8th row, use substitution to substitute \( AB' \) for \( BA \) and \( B'D \) for \( BD' \).

In the 9th row, use the addition property to combine like terms.

In the 10th row, use the distributive property to factor out a common factor of 2.

In the 11th row, use the segment addition postulate.

In the 12th row, use substitution with row 10 and row 11.

In the 13th row, substitute \( x \) for \( AD \).

**Proof:**

**Statements (Reasons):**

1. A reflection in line \( p \) maps \( \overline{BC} \) to \( \overline{B'C'} \); a reflection in line \( q \) maps \( \overline{B'C'} \) to \( \overline{B''C''} \); \( p \parallel q \); \( x \) is the distance between \( p \) and \( q \). (Given)

2. \( p \) is the perpendicular bisector of \( \overline{BB'} \), and \( q \) is the perpendicular bisector of \( \overline{B'B''} \). (Definition of \( \perp \) bisector)

3. \( BB' + B'B'' = BB'' \) (Seg. Add. Post.)

4. \( BB'' \perp p \), \( BB'' \perp q \) (A line perpendicular to a portion of a segment is perpendicular to the whole segment.)

5. \( BA \cong AB'; B'D \cong BD'' \) (Def. of refl.)
3-4 Compositions of Transformations

6. \( BA = AB' \); \( BD = DB'' \) (Def. of \( \cong \))
7. \( BA + AB' + BD + DB'' = BB'' \) (Seg. Add. Post.)
8. \( AB' + AB' + BD + BD = BB'' \) (Subs.)
9. \( 2AB' + 2BD = BB'' \) (Add. Prop.)
10. \( 2(AB' + BD) = BB'' \) (Dist. Prop.)
11. \( AB' + BD = AD \) (Seg. Add. Post.)
12. \( 2AD = BB'' \) (Subs.)
13. \( 2x = BB'' \) (Subs.)

35. PROOF Write a paragraph proof of Theorem 3.3.

Given: Lines \( \ell \) and \( m \) intersect at point \( P \). \( A \) is any point on \( \ell \) or \( m \).
Prove: a. If you reflect point \( A \) in \( m \), and then reflect \( A' \) in \( \ell \), \( A'' \) is the image of \( A \) after a rotation about \( P \).
   b. \( m\angle APA'' = 2(m\angle SPR) \)

SOLUTION:
Walk through the proof step by step. Review the given statements and use the properties you have learned about reflections and translations to walk through the proof.

Proof: We are given that \( \ell \) and \( m \) intersect at point \( P \), not on \( \ell \) or \( m \). Reflect \( A \) over line \( m \) to \( A' \) and reflect \( A' \) over line \( \ell \) to \( A'' \). By the definition of reflection, \( m \) is the perpendicular bisector of \( A'A'' \). \( \ell \) is the perpendicular bisector of \( A'A'' \) by the definition of a perpendicular bisector. Any two points there is exactly one line, so we can draw segments, \( A'A'' \), \( A'A'' \), and \( A'A'' \). Angle \( APR \), angle \( A'PS \) and angle \( A''SP \) are right angles by the definition of perpendicular bisectors. \( \overline{AR} \parallel \overline{AP} \) and \( \overline{SP} \parallel \overline{SP} \) by the Reflexive Property. \( \angle ARP \equiv \angle A'RP \) and \( \angle A'RP \equiv \angle A''SP \) by the SAS Converse Postulate. Using CPCTC, \( \overline{AP} \equiv \overline{A''P} \), \( \overline{AP} \equiv \overline{A''P} \), an \( \overline{AP} \equiv \overline{A''P} \) by the Transitive Property.

By the definition of a rotation, \( A'' \) is the image of \( A \) at point \( P \). Also using CPCTC,
\[ \angle APR \equiv \angle A''PR \] and \[ \angle A'PS \equiv \angle A''PS \].

By the definition of congruence,
\[ m\angle APR = m\angle A'PR \] and \[ m\angle A'PS = m\angle A''PS \].
\[ m\angle A''PS + m\angle A'PR = m\angle SPR \] by the Angle Addition Postulate.

Simplifies to \( 2(m\angle A'PR + m\angle A''PS) = m\angle APA'' \). By substitution, \( 2(m\angle SPR) = m\angle APA'' \)

SOLUTION:
We are given that \( \ell \) and \( m \) intersect at point \( P \), not on \( \ell \) or \( m \). Reflect \( A \) over line \( m \) to \( A' \) and reflect \( A' \) over line \( \ell \) to \( A'' \). By the definition of reflection, \( m \) is the perpendicular bisector of \( A'A'' \). \( \ell \) is the perpendicular bisector of \( A'A'' \) by the definition of a perpendicular bisector. Any two points there is exactly one line, so we can draw segments, \( A'A'' \), \( A'A'' \), and \( A'A'' \). Angle \( APR \), angle \( A'PS \) and angle \( A''SP \) are right angles by the definition of perpendicular bisectors. \( \overline{AR} \parallel \overline{AP} \) and \( \overline{SP} \parallel \overline{SP} \) by the Reflexive Property. \( \angle ARP \equiv \angle A'RP \) and \( \angle A'RP \equiv \angle A''SP \) by the SAS Converse Postulate. Using CPCTC, \( \overline{AP} \equiv \overline{A''P} \), \( \overline{AP} \equiv \overline{A''P} \), an \( \overline{AP} \equiv \overline{A''P} \) by the Transitive Property.

36. ERROR ANALYSIS Daniel and Lolita are translating \( \Delta XYZ \) along \( \langle 2, 2 \rangle \) and reflecting it in the line \( y = 2 \). Daniel says that the transformation is a glide reflection. Lolita disagrees and says that the transformation is a composition of transformations. Is either of them correct? Explain your reasoning.
### 3-4 Compositions of Transformations

**SOLUTION:**

**Step 1:** Translate along \((2, 2)\).

\[(x, y) \rightarrow (x + 2, y + 2)\]

- \(X'(-3, -2) \rightarrow X'(0, 0)\)
- \(Y(0, 0) \rightarrow Y'(2, 2)\)
- \(Z(1, -2) \rightarrow Z'(3, 0)\)

**Step 2:** Reflect over the \(y = 2\) line.

- 2 units from \(y = 2\): \(X''(-1, 0) \rightarrow X''(-1, 4)\)
- on \(y = 2\): \(Y''(2, 2) \rightarrow Y''(2, 0)\)
- 2 units from \(y = 2\): \(Z''(3, 0) \rightarrow Z''(3, 4)\)

The transformation is a composition of translation and a reflection, so it is a composition of transformations. Therefore, Lolita is correct. The line \(y = 2\) is not parallel to the vector \((2, 2)\), thus, the transformation cannot be a glide reflection.

**ANSWER:**

Lolita is correct; Sample answer: Since the line \(y = 2\) is not parallel to the vector \((2, 2)\), the transformation cannot be a glide reflection. It is a composition of translation and a reflection, so it is a composition of transformations.

37. **WRITING IN MATH** Do any points remain invariant under glide reflections? Under compositions of transformations? Explain.

**SOLUTION:**

Invariant points map onto themselves.

No points remain invariant under glide reflections. This is because all of the points are translated along a vector.

Consider the glide reflection shown below. There are no invariant points.

It is possible for points to remain invariant under a composition of transformations. There may be invariant points when a figure is rotated and reflected, rotated twice, or reflected twice.

Two 180° rotations will bring the image back to the original image.

Two reflections about the same line will do the same.
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A rotation and reflection can also bring the image back to the original image. Triangle $ABC$ is rotated 240 degrees about the origin and then reflected twice; in the $x$-axis and in the dashed line. The final image maps to the original image.

**ANSWER:**
Sample answer: No; there are no invariant points in a glide reflection because all of the points are translated along a vector. Perhaps for compositions of transformations, there may be invariant points when a figure is rotated and reflected, rotated twice, or reflected twice.

38. **CHALLENGE** If $PQRS$ is translated along $(3,-2)$, reflected in $y = -1$, and rotated $90^\circ$ about the origin, what are the coordinates of $P''Q''R''S'''$?

**SOLUTION:**
The coordinates of $P, Q, R,$ and $S$ are $(-5,1), (-2,2), (0,-1),$ and $(-3,-2)$ respectively.

**Step 1:** Translation along $(3,-2)$.

$$(x,y) \Rightarrow (x+3,y-2)$$
$$P(-5,1) \Rightarrow P'(-2,-1)$$
$$Q(-2,2) \Rightarrow Q'(1,0)$$
$$R(0,-1) \Rightarrow R'(3,-3)$$
$$S(-3,-2) \Rightarrow S'(0,-4)$$

**Step 2:** Reflection in $y = -1$.

$$P'(1,1) \Rightarrow P'''(3,-1)$$
$$Q'(1,0) \Rightarrow Q'''(1,-2)$$
$$R'(3,-3) \Rightarrow R'''(3,1)$$
$$S'(0,-4) \Rightarrow S'''(0,2)$$

**Step 3:** Rotation $90^\circ$ about the origin.

$$(x,y) \Rightarrow (-y,x)$$
$$P'''(1,-2) \Rightarrow P''''(1,-2)$$
$$Q'''(1,-2) \Rightarrow Q''''(2,1)$$
$$R''''(3,1) \Rightarrow R''''(-1,3)$$
$$S'''(0,2) \Rightarrow S''''(-2,0)$$

The coordinates of $P''''Q''''R''''S''''$ are $P''''(1,-2), Q''''(2,1), R''''(-1,3), S''''(-2,0)$.

**ANSWER:**

$P''''(1,-2), Q''''(2,1), R''''(-1,3), S''''(-2,0)$
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39. **CONSTRUCT ARGUMENTS** If an image is to be reflected in the line $y = x$ and the x-axis, does the order of the reflections affect the final image? Explain.

**SOLUTION:**
Yes, the order of the reflections affects the final image if an image is to be reflected in the line $y = x$ and the x-axis.

If a segment with endpoints $(a, b)$ and $(c, d)$ is to be reflected about the x-axis, the coordinates of the endpoints of the reflected image are $(a, -b)$ and $(c, -d)$. If the segment is then reflected about the line $y = x$, the coordinates of the endpoints of the final image are $(b, -a)$ and $(d, -c)$.

If the original image is first reflected about $y = x$, the coordinates of the endpoints of the reflected image are $(b, a)$ and $(d, c)$. If the segment is then reflected about the x-axis, the coordinates of the endpoints of the final image are $(b, a)$ and $(d, c)$.

40. **OPEN-ENDED** Identify a glide reflection or sequence of transformations that can be used to transform $\triangle ABC$ to $\triangle DEF$.

**SOLUTION:**
First study triangles $ABC$ and $DEF$. $BC$ and $EF$ have the same slope. Angles $A$ and $D$ are right angles. So, map point $A$ to point $D$, point $B$ to point $E$, and point $C$ to point $F$.

The x-coordinates of points $C$ and $F$ are two units apart. The y-coordinates of $B$ and $E$ are 4 units apart. Translate $ABC$ along the vector $<0, -4>$ so each pair of corresponding points lie on the same horizontal line. Next, reflect points lie on the same horizontal line. Next, reflect $\triangle ABC$ in the line $x = -1$ to form $\triangle DEF$.

**ANSWER:**
Sample answer: $\triangle ABC$ can be translated along $\{0, -4\}$ and reflected in $x = -1$ to form $\triangle DEF$. 

**ANSWER:**
Yes; sample answer: If a segment with endpoints $(a, b)$ and $(c, d)$ is to be reflected about the x-axis, the coordinates of the endpoints of the reflected image are $(a, -b)$ and $(c, -d)$. If the segment is then reflected about the line $y = x$, the coordinates of the endpoints of the final image are $(b, -a)$ and $(d, -c)$.

If the original image is first reflected about $y = x$, the coordinates of the endpoints of the reflected image are $(a, -b)$ and $(c, -d)$. If the segment is then reflected about the x-axis, the coordinates of the endpoints of the final image are $(b, -a)$ and $(d, -c)$.

If the original image is first reflected about the x-axis, the coordinates of the endpoints of the reflected image are $(a, -b)$ and $(c, -d)$. If the segment is then reflected about the line $y = x$, the coordinates of the endpoints of the final image are $(b, -a)$ and $(d, -c)$. 

If the original image is first reflected about the line $y = x$, the coordinates of the endpoints of the reflected image are $(b, a)$ and $(d, c)$. If the segment is then reflected about the x-axis, the coordinates of the endpoints of the final image are $(b, a)$ and $(d, c)$.
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41. **ARGUMENTS** When two rotations are performed order of the rotations *sometimes, always, or never* affect image? Explain.

**SOLUTION:**
When two rotations are performed on a single image, rotations *sometimes* affect the location of the final image. When two rotations are performed on a single image, they do not affect the final image when the two rotation points are the same. For example, if $\triangle ABC$ is rotated $45^\circ$ clockwise about the origin, $\triangle A'B'C'$ is the same as the first rotated $60^\circ$ clockwise about the origin and then $r$ the origin.

If $\triangle ABC$ is rotated $45^\circ$ clockwise about the origin, then rotated $60^\circ$ clockwise about the origin, $\triangle A''B''C''$ is different than the original.

So, the order of the rotations sometimes affects the final image.

**ANSWER:**
Sometimes; sample answer: When two rotations are performed on the same image, the order of the rotations does not affect the final image because the rotations are centered at the same point.

42. **WRITING IN MATH** Compare and contrast glide reflections and compositions of transformations.

**SOLUTION:**
Sample answer: Glide reflections are compositions of transformations. But not all compositions of transformations are glide reflections. Triangle $JKL$ is translated and then reflected in the $y$-axis. This is a glide reflection.

Rotations can be included in compositions of transformations but not glide reflections. Segment $A$ is reflected in line $K$ and then reflected in line $\ell$. This composition of transformations can be described by a rotation.

Translations and reflections can both be used in compositions of transformations, but only make up a glide reflection when a figure is translated along a vector and then reflected in a line parallel to that vector.

**ANSWER:**
Sample answer: Glide reflections are compositions of transformations. But not all compositions of transformations are glide reflections. Rotations can be included in compositions of transformations but not glide reflections. Translations and reflections can both be used in compositions of transformations, but only make up a glide reflection when a figure is translated along a vector and then reflected in a line parallel to that vector.
43. Which composition of transformations maps \( \overline{JK} \) to \( \overline{LM} \)?

- A: rotation 90° about the origin and translation along \( \langle 0, -1 \rangle \)
- B: reflection in \( y = -1 \) and translation along \( \langle -1, -2 \rangle \)
- C: translation along \( \langle 0, -1 \rangle \) and rotation 90° about the origin
- D: translation along \( \langle -1, 1 \rangle \) and reflection in \( y \)-axis

**SOLUTION:**
Consider each option (A-D) visually to determine which could possibly be the solution. For option C, consider the shift in the coordinates:
- Translation of \( \overline{JK} \) along \( \langle 0, -1 \rangle \):
  \( (x, y) \rightarrow (x, y - 1) \)
  \( J(1, 3) \rightarrow J'(1, 2) \)
  \( K(4, 1) \rightarrow K'(4, 0) \)
- Rotation of \( \overline{JK} \) 90° about the origin:
  \( (x, y) \rightarrow (-y, x) \)
  \( J'(1, 2) \rightarrow J''((-2, 1)) \)
  \( K'(4, 0) \rightarrow K''((0, 4)) \)

The coordinates for \( J'' \) and \( K'' \) match the coordinates for \( L \) and \( M \). The correct choice is C.

**ANSWER:** C

44. Miguel draws \( \triangle ABC \) as shown. Then he rotates the triangle 180° about the origin and reflects the image in the \( y \)-axis.

Which of the following points is a vertex of the final image of the triangle?

- A: \((-1, -3)\)
- B: \((3, 1)\)
- C: \((1, -3)\)
- D: \((3, -2)\)
- E: \((-2, -2)\)

**SOLUTION:**
Take each vertex of \( \triangle ABC \) through the given sequence of transformations:
- 180° rotation about the origin, \( (x, y) \rightarrow (-x, -y) \)
  \( A(-3, 1) \rightarrow A'(3, -1) \)
  \( B(-2, 3) \rightarrow B'(2, -3) \)
  \( C(-1, 3) \rightarrow C'(1, -3) \)

Then, a reflection across the \( y \)-axis.
  \( (x, y) \rightarrow (-x, y) \)
  \( A'(3, -1) \rightarrow A''( -3, -1) \)
  \( B'(2, -3) \rightarrow B''((-2, -3)) \)
  \( C'(1, -3) \rightarrow C''((-1, -3)) \)

The correct choice is A.

**ANSWER:**
A

45. **MULTI-STEP** Jada is creating a design on a sheet of graph paper by drawing and transforming a triangle. She draws triangle \( PQR \) with vertices \( P(-4, 1) \), \( Q(-4, -3) \), and \( R(-2, -2) \) as shown.

a. Jada first translates triangle \( PQR \) along \( \langle 5, 2 \rangle \).

Given the translation \( (x, y) \rightarrow (x + 5, y + 2) \), what
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are the coordinates of triangle $P'Q'R''$?

$P' = \ldots$

$Q' = \ldots$

$R' = \ldots$

b. Jada next reflects the image triangle $P'Q'R'$ in the $x$-axis to get the final image triangle $P''Q''R''$. Draw the final image on a sheet of graph paper. What are the coordinates of the vertices of triangle $P''Q''R''$?

$P'' = \ldots$

$Q'' = \ldots$

$R'' = \ldots$

c. Which of the following points lies in the interior of triangle $P''Q''R''$?

A (2, 0)

B (–2, 0)

C (2, 3)

D (2, –3)

E (3, –2)

F (3, 2)

**SOLUTION:**

**MULTI-STEP** Jada is creating a design on a sheet of graph paper by drawing and transforming a triangle. She draws triangle $PQR$ with vertices $P(-4, 1), Q(-4, -3)$, and $R(-2, -2)$ as shown.

![Graph Paper with Triangle](image)

a. Translates the vertices $P(-4, 1), Q(-4, -3)$, and $R(-2, -2)$ using the translation $(x, y) \rightarrow (x + 5, y + 2)$.

$P' = (-4 + 5, 1 + 2) = (1, 3)$

$Q' = (-4 + 5, -3 + 2) = (1, -1)$

$R' = (-2 + 5, -2 + 2) = (3, 0)$

b. Reflects the image triangle $P'Q'R'$ in the $x$-axis using the translation $(x, y) \rightarrow (x, -y)$.

$P'' = (1, -3)$

$Q'' = (1, 1)$

$R'' = (3, 0)$

Draw this on graph paper so you can answer part c to find which point lies in the interior of this final image.

The correct answer is A, because the final triangle is in quadrants I and IV, but gets no higher than 1 in the $y$-coordinate, so C and F are not possible answers.

**ANSWER:**

a. $P' = (-4 + 5, 1 + 2) = (1, 3)$

$Q' = (-4 + 5, -3 + 2) = (1, -1)$

$R' = (-2 + 5, -2 + 2) = (3, 0)$

b. $P'' = (1, -3)$

$Q'' = (1, 1)$

$R'' = (3, 0)$

c. A